JOURNAL OF APPROXIMATION THEORY 6, 439–465 (1972)

Polynomial Splines and a Fundamental Eigenvalue Problem for Polynomials

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Communicated by Oved Shisha

Received October 30, 1970

DEDICATED TO PROFESSOR J. L. WALSH ON THE OCCASION OF HIS 75TH BIRTHDAY

INTRODUCTION

It has been a significant development in the computational theory of cubic splines that all such splines on uniform meshes can be generated in a very simple manner from a special cubic segment [1, 2, 3]. It seems clear that it should be possible to extend this result to general polynomial splines (cf. Ref. [4]) and this is indeed the case. For polynomial splines of degree $2n + 1$ on uniform meshes, there are precisely n special polynomial segments on $[0, 1]$ of degree $2n + 1$ which serve to generate all such splines.

These results are derived from the consideration of a simple and familiar eigenvalue problem. The resulting structure sheds new light upon the intrinsic nature of splines, and considerably more of the relationship between polynomial splines and the Hille polynomials now becomes apparent. For the most part the results are unexpectedly simple.

A FUNDAMENTAL EIGENVALUE PROBLEM FOR POLYNOMIALS

We consider a polynomial $p(\theta)$ of degree m with $p(0) = p(1) = 0$. Such a polynomial is determined by the values of its first $m - 1$ deratives at $\theta = 0$, namely, by

$$p_0 = \{ p'(0), p''(0),..., p^{(m-1)}(0)\}.$$

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Let T be the transformation taking p_0 into

$$p_1 = \{p'(1), p''(1), \ldots, p^{(m-1)}(1)\}:$$
$$p_1 = Tp_0 .$$

If we require $p_j(0) = p_j(1) = 0$ and $p_j^{(k)}(0) = \delta_{jk}$ (Kronecker delta; $j = 1, \ldots, m-1$; $k = 1, \ldots, m-1$), $p_j(\theta)$ a polynomial of degree m, then

$$p(\theta) = p'(0)\, p_1(\theta) + p''(0)\, p_2(\theta) + \cdots + p^{(m-1)}(0)\, p_{m-1}(\theta),$$

so that

$$T = \begin{bmatrix} p_1'(1) & p_2'(1) & \cdots & p_{m-1}'(1) \\ p_1''(1) & p_2''(1) & \cdots & p_{m-1}''(1) \\ \vdots & \vdots & & \vdots \\ p_1^{(m-1)}(1) & p_2^{(m-1)}(1) & \cdots & p_{m-1}^{(m-1)}(1) \end{bmatrix}. \tag{1}$$

Note that

$$p_j(\theta) = (\theta^j - \theta^m)/j!$$

so that

$$\begin{aligned} p_j^{(k)}(1) &= 1/(j-k)! - m^{(k)}/j!, \qquad & k \leqslant j, \\ &= -m^{(k)}/j!, & k > j, \end{aligned} \tag{2}$$

where $m^{(k)} = m(m-1) \cdots (m-k+1)$.

The eigenvalues and eigenvectors of T are of particular interest. To express these, we introduce here the spline characteristic polynomials [5, 6] $\delta_k(\lambda)$:

$$\begin{aligned} \delta_1(\lambda) &= 1, \\ \delta_2(\lambda) &= 1 + \lambda, \\ \delta_3(\lambda) &= 1 + 4\lambda + \lambda^2, \\ \delta_4(\lambda) &= 1 + 11\lambda + 11\lambda^2 + \lambda^3, \\ \delta_5(\lambda) &= 1 + 26\lambda + 66\lambda^2 + 26\lambda^3 + \lambda^4, \\ &\vdots \end{aligned}$$

We have as a recursive relation [Ref. 5] for these polynomials

$$\begin{aligned} \delta_m &= m\lambda\delta_{m-1} + (1-\lambda)(\lambda\delta_{m-1})' \\ &= [(m-1)\lambda + 1]\, \delta_{m-1} + \lambda(1-\lambda)\, \delta_{m-1}' . \end{aligned} \tag{3}$$

We note [7, p. 47] that $\delta_m(\lambda) \equiv P_m(\lambda)/\lambda$, where $P_m(\lambda)$ is the Hille polynomial defined by

$$(d/d \log \lambda)^m\, 1/(1-\lambda) = P_m(\lambda)/(1-\lambda)^{m+1}. \tag{4}$$

The zeros of $\delta_m(\lambda)$ are real, negative, and distinct [5]. -1 is a zero iff m is even. Zeros distinct from -1 occur in reciprocal pairs.
We obtain the following fundamental result:

THEOREM 1. *The eigenvalues of the matrix T are the zeros $\lambda_1, \lambda_2, ..., \lambda_{m-1}$ of $\delta_m(\lambda)$. The eigenvectors are* ($j = 1, 2, ..., m-1$):

$$\begin{pmatrix} \delta_{m-1}(\lambda_j) \\ -(m-1)(1-\lambda_j)\,\delta_{m-2}(\lambda_j) \\ (m-1)^{(2)}(1-\lambda_j)^2\,\delta_{m-3}(\lambda_j) \\ \vdots \\ (-1)^{m-2}(m-1)^{(m-2)}(1-\lambda_j)^{m-2}\,\delta_1(\lambda_j) \end{pmatrix}. \tag{5}$$

Proof. We form the scalar product of the k-th row of the matrix T with the vector (5), obtaining

$$p_1^{(k)}(1)\cdot\delta_{m-1}(\lambda_j) - p_2^{(k)}(1)(m-1)^{(1)}(1-\lambda_j)\,\delta_{m-2}(\lambda_j) \\ + \cdots + (-1)^{m-2}\,p_{m-1}^{(k)}(1)(m-1)^{(m-2)}(1-\lambda_j)^{m-2}\,\delta_1(\lambda_j).$$

In view of (2), we need to show that

$$\begin{aligned} &\frac{(m)^{(k)}}{1!}\delta_{m-1}(\lambda_j) - \frac{(m)^{(k)}}{2!}(m-1)(1-\lambda_j)\,\delta_{m-2}(\lambda_j) \\ &\quad + \cdots + (-1)^k\frac{(m)^{(k)}}{(k-1)!}(m-1)^{(k-2)}(1-\lambda_j)^{k-2}\,\delta_{m-k+1}(\lambda_j) \\ &\quad + (-1)^k\left[1 - \frac{(m)^{(k)}}{k!}\right](m-1)^{(k-1)}(1-\lambda_j)^{k-1}\,\delta_{m-k}(\lambda_j) \\ &\quad + \cdots + (-1)^{m-1}\left[\frac{1}{(m-k-1)!} - \frac{(m)^{(k)}}{(m-1)!}\right] \\ &\quad \times (m-1)^{(m-2)}(1-\lambda_j)^{m-2}\,\delta_1(\lambda_j) \\ &= \lambda_j(-1)^k(m-1)^{(k-1)}(1-\lambda_j)^{k-1}\,\delta_{m-k}(\lambda_j). \end{aligned} \tag{6}$$

For this purpose we derive several basic properties of the spline characteristic polynomials. We set

$$\mathscr{D} = d/d\log\lambda$$

and use (4) to obtain directly

$$\mathscr{D}\,\lambda\delta_n(\lambda)/(1-\lambda)^{n+1} = \lambda\delta_{n+1}(\lambda)/(1-\lambda)^{n+2}. \tag{7}$$

By means of this, the following identity is established:

LEMMA 1.

$$\delta_m(\lambda) = \binom{m}{1}\delta_{m-1}(\lambda) - \binom{m}{2}(1-\lambda)\,\delta_{m-2}(\lambda) + \cdots + (-1)^{m-2}\binom{m}{m-1}(1-\lambda)^{m-2}\,\delta_1(\lambda) + (-1)^{m-1}(1-\lambda)^{m-1}. \tag{8}$$

Proof. We proceed by induction. The identity (8) is clearly valid for $m = 1$. If we assume its validity for m, we obtain ($\lambda \neq 1$) the relation

$$\frac{\lambda\delta_m(\lambda)}{(1-\lambda)^{m+1}} = \binom{m}{1}\frac{1}{1-\lambda}\frac{\lambda\delta_{m-1}(\lambda)}{(1-\lambda)^m} - \binom{m}{2}\frac{1}{1-\lambda}\frac{\lambda\delta_{m-2}(\lambda)}{(1-\lambda)^{m-1}} + \cdots + (-1)^{m-2}\binom{m}{m-1}\frac{1}{1-\lambda}\frac{\lambda\delta_1(\lambda)}{(1-\lambda)^2} + (-1)^{m-1}\frac{\lambda}{(1-\lambda)^2}.$$

Applying the operator $\mathscr{D}$ now gives by (7) ($\delta_0(\lambda) \equiv 1$):

$$\frac{\lambda\delta_{m+1}(\lambda)}{(1-\lambda)^{m+2}} = \sum_{j=1}^{m}\binom{m}{j}(-1)^{j-1}\mathscr{D}\left\{\frac{1}{1-\lambda}\frac{\lambda\delta_{m-j}(\lambda)}{(1-\lambda)^{m-j+1}}\right\}$$

$$= \sum_{j=1}^{m}\binom{m}{j}(-1)^{j-1} \times \left\{\frac{\lambda}{(1-\lambda)^2}\frac{\lambda\delta_{m-j}(\lambda)}{(1-\lambda)^{m-j+1}} + \frac{1}{1-\lambda}\frac{\lambda\delta_{m-j+1}(\lambda)}{(1-\lambda)^{m-j+2}}\right\}.$$

Rearranging and dividing by $\lambda/(1-\lambda)^{m+2}$, we have

$$\delta_{m+1}(\lambda) = m\delta_m(\lambda) + \sum_{j=1}^{m-1}(-1)^{j-1}(1-\lambda)^j\,\delta_{m-j}(\lambda)\left\{\binom{m}{j}\frac{\lambda}{1-\lambda} - \binom{m}{j-1}\right\} + (-1)^{m-1}\lambda(1-\lambda)^{m-1}\,\delta_0(\lambda).$$

In the right-hand member, add and subtract the left- and right-hand members of (8), the inductive assumption, obtaining

$$\delta_{m+1}(\lambda) = (m+1)\,\delta_m(\lambda) + \sum_{j=1}^{m-1}(-1)^{j-1}(1-\lambda)^j\,\delta_{m-j}(\lambda) \cdot \left\{\binom{m}{j}\frac{\lambda}{1-\lambda} - \binom{m}{j-1} - \binom{m}{j}\frac{1}{1-\lambda}\right\} + (-1)^m(1-\lambda)^m\,\delta_0(\lambda).$$

Inasmuch as the term in braces is equal to $\binom{m+1}{j}$, the induction proof is complete.

We require also the following identity which is a consequence of the preceding lemma:

LEMMA 2.

$$\delta_m(\lambda) = \left[\lambda + \binom{m-1}{1}\right] \delta_{m-1}(\lambda) - \binom{m-1}{2} (1-\lambda)\, \delta_{m-2}(\lambda)$$
$$+ \binom{m-1}{3} (1-\lambda)^2 \delta_{m-3}(\lambda) + \cdots + (-1)^{m-2} (1-\lambda)^{m-2} \delta_1(\lambda). \quad (9)$$

Proof. By Lemma 1, the difference between the left- and right-hand members of (9) is

$$\left\{\binom{m}{1} \delta_{m-1}(\lambda) - \binom{m}{2} (1-\lambda)\, \delta_{m-2}(\lambda) + \binom{m}{3} (1-\lambda)^2 \delta_{m-3}(\lambda)\right.$$
$$\left. - \cdots + (-1)^{m-2} \binom{m}{m-1} (1-\lambda)^{m-2} \delta_1(\lambda) + (-1)^{m-1} (1-\lambda)^{m-1}\right\}$$
$$- \left\{\left[\lambda + \binom{m-1}{1}\right] \delta_{m-1}(\lambda) - \binom{m-1}{2} (1-\lambda)\, \delta_{m-2}(\lambda)\right.$$
$$\left. + \binom{m-1}{3} (1-\lambda)^2 \delta_{m-3}(\lambda) + \cdots + (-1)^{m-2} (1-\lambda)^{m-2} \delta_1(\lambda)\right\}$$
$$= (1-\lambda) \left\{\delta_{m-1}(\lambda) - \binom{m-1}{2} (1-\lambda)\, \delta_{m-2}(\lambda)\right.$$
$$\left. + \cdots + (-1)^{m-2} \binom{m-1}{m-2} (1-\lambda)^{m-3} \delta_1(\lambda) + (-1)^{m-1} (1-\lambda)^{m-2}\right\}.$$

which is zero.

The validity of (6) now follows directly: The left-hand member of (6) is equal to

$$(m-1)^{(k-1)} \left\{\binom{m}{1} \delta_{m-1}(\lambda_j) - \binom{m}{2} (1-\lambda_j)\, \delta_{m-2}(\lambda_j)\right.$$
$$\left. + \cdots + (-1)^{m-2} \binom{m}{m-1} (1-\lambda_j)^{m-2} \delta_1(\lambda_j) + (-1)^{m-1} (1-\lambda_j)^{m-1}\right\}$$
$$+ (-1)^k (m-1)^{(k-1)} (1-\lambda_j)^{k-1}$$
$$\times \left\{\delta_{m-k}(\lambda_j) - \binom{m-k}{1} (1-\lambda_j)\, \delta_{m-k-1}(\lambda_j)\right.$$
$$+ \cdots + (-1)^{m-k-1} \binom{m-k}{m-k-1} (1-\lambda_j)^{m-k-1} \delta_1(\lambda_j)$$
$$\left. + (-1)^{m-k} (1-\lambda_j)^{m-k}\right\}.$$

The first term in braces is equal to $\delta_m(\lambda_j)$ which is equal to zero. The second term in braces becomes

$$\lambda_j\delta_{m-k}(\lambda_j) + (1-\lambda_j)\left[\delta_{m-k}(\lambda_j) - \binom{m-k}{1}\delta_{m-k-1}(\lambda_j) + \cdots + (-1)^{m-k-1}\binom{m-k}{m-k-1}(1-\lambda_j)^{m-k-2}\,\delta_1(\lambda_j) + (-1)^{m-k}(1-\lambda_j)^{m-k-1}\right],$$

where the bracketed term vanishes by Lemma 1. This completes the proof of Theorem 1. Inasmuch as the eigenvalues are distinct and nonzero, the eigenvectors given are independent.

We renumber these eigenvalues λ_j in decreasing order:

$$\lambda_{m-1} < \lambda_{m-2} < \cdots \lambda_1 < 0.$$

When $m = 2n + 1$ is odd,

$$\lambda j \cdot \lambda_{2n+1-j} = 1.$$

We wish to note in passing the work of Schoenberg [8] where this eigenvalue problem is considered from another point of view.

The Cardinal Splines $C_j(\theta)$

The j-th eigenvector of (1) determines a spline of degree m on $[0, \infty)$ in the following sense. Specify the polynomial $s_j(\theta)$ by

$$s_j(0) = s_j(1) = 0,$$
$$s_j^{(p)}(0) = (-1)^{p-1}(m-1)^{(p-1)}(1-\lambda_j)^{p-1}\,\delta_{m-p}(\lambda_j), \qquad p = 1, 2, ..., m-1.$$

We restrict this polynomial to [0, 1] and define the spline $C_j(\theta)$ on $[0, \infty)$ by

$$C_j(\theta) \in C^{m-1}[0, \infty),$$
$$C_j(\theta) = s_j(\theta), \qquad 0 \leqslant \theta \leqslant 1,$$
$$C_j(\theta + 1) = \lambda_j C_j(\theta).$$

For $|\lambda_j| < 1$, $C_j(\theta) \to 0$ as $\theta \to \infty$. For $m = 2n + 1$, there are n such splines damped to the right, and n splines increasing in amplitude to the right.

We normalize $s_j(\theta)$ to have $s_j'(0) = 1$ and modify the $C_j(\theta)$ accordingly. Thus we use instead

$$
\begin{aligned}
s_j(\theta) &= \theta - \frac{m-1}{2}(1-\lambda_j)\frac{\delta_{m-2}(\lambda_j)}{\delta_{m-1}(\lambda_j)}\theta^2 \\
&\quad + \frac{(m-1)(m-2)}{2\cdot 3}(1-\lambda_j)^2\frac{\delta_{m-3}(\lambda_j)}{\delta_{m-1}(\lambda_j)}\theta^3 \\
&\quad - \cdots + (-1)^{m-2}\frac{(m-1)^{(m-2)}}{(m-1)!}(1-\lambda_j)^{m-2}\frac{\delta_1(\lambda_j)}{\delta_{m-1}(\lambda_j)}\theta^{m-1} + \alpha_j\theta^m/m \\
&= \frac{1}{m}\left[\binom{m}{1}\theta - \binom{m}{2}(1-\lambda_j)\frac{\delta_{m-2}(\lambda_j)}{\delta_{m-1}(\lambda_j)}\theta^2\right. \\
&\quad \left. + \cdots + (-1)^{m-2}\binom{m}{m-1}(1-\lambda_j)\frac{\delta_1(\lambda_j)}{\delta_{m-1}(\lambda_j)}\theta^{m-1} + \alpha_j\theta^m\right].
\end{aligned}
\tag{10}
$$

Since $s_j(1) = 0$ and $\delta_m(\lambda_j) = 0$, Lemma 1 gives

$$\alpha_j = (-1)^{m-1}(1-\lambda_j)^{m-1}/\delta_{m-1}(\lambda_j) \tag{11}$$

[By (3) it is seen that $\delta_{m-1}(\lambda_j) \neq 0$, for $\delta_{m-1}(\lambda)$ cannot have a multiple zero.]

Let us define

$$r_j = -\tfrac{1}{2}(\lambda_j + 1/\lambda_j), \qquad j = 1, 2,\dots, n, \tag{12}$$

and for even and odd k express $\delta_k(\lambda_j)$ as follows:

$$\delta_{2p}(\lambda_j) \equiv \lambda_j^{2p-1}\delta_{2p}(1/\lambda_j) \equiv (1+\lambda_j)\,\lambda_j^{p-1}u_{p-1}(r_j), \tag{13}$$

$$\delta_{2p+1}(\lambda_j) \equiv \lambda_j^{2p}\delta_{2p+1}(1/\lambda_j) \equiv \lambda_j^{p}v_p(r_j). \tag{14}$$

Moreover,

$$s_{m-j}(1-\theta) = -(1/\lambda_j)\,s_j(\theta), \tag{15}$$

for both sides have their first $m-1$ derivatives multiplied by λ_j as θ goes from 0 to 1, have equal first derivatives at $\theta = 0$, and hence take on the unique set of first $(m-1)$ derivative values at $\theta = 0$ associated with λ_j in (5).

From (15) and the definition of $C_j(\theta)$ we have

$$C_{m-j}(1-\theta) = -(1/\lambda_j)\,C_j(\theta), \tag{16}$$

or, alternatively,

$$C_{m-j}(\theta) = -C_j(\theta). \tag{17}$$

THE CARDINAL SPLINE $A(\theta)$

We restrict our attention henceforth to the case of odd m, $m = 2n + 1$. The spline $A(\theta)$ is of degree $2n + 1$, with knots at the integer points, and

$$A(\theta) \in C^{2n}(-\infty, \infty), \qquad \lim_{\theta \to \infty} A(\theta) = 0,$$

$$A(0) = 1, \qquad A(k) = 0\,(k, \pm 1, \pm 2,...), \qquad A(-\theta) = A(\theta).$$

We require a polynomial segment $q(\theta)$ on $[0, 1]$ with odd-order derivatives through the $(2n - 1)$st vanishing at $\theta = 0$ and for which $q(0) = 1$, $q(1) = 0$:

$$q(\theta) = 1 + (a_1 - 1)\,\theta^2 + (a_2 - a_1)\,\theta^4 + \cdots + (a_n - a_{n-1})\,\theta^{2n} - a_n\theta^{2n+1} \tag{18}$$

We must choose $a_1, a_2, ..., a_n$ and obtain a suitable linear combination

$$b_1C_1(\theta) + b_2C_2(\theta) + \cdots + b_nC_n(\theta), \tag{19}$$

so that (18) and (19) have their first $2n$ derivatives equal at $\theta = 1$. Then (19) gives $A(\theta)$ on $[1, \infty)$.

Now

$$q(\theta) - (1 - \theta)^{2n+1}$$

vanishes at 0 and 1, has the same first $2n$ derivatives at $\theta = 1$ as $q(\theta)$, and so it coincides on $[0, 1]$ with (19).

Thus we choose $b_1, b_2, ..., b_n$ so as to give zero for $q^{(p)}(0)$, $p = 1, 3, ..., 2n - 1$ and the quantities a_j need not be determined:

$$\begin{aligned} b_1 + \cdots + b_n &= (2n+1)^{(1)}, \\ b_1(2n)(2n-1)\frac{\delta_{2n-2}(\lambda_1)}{\delta_{2n}(\lambda_1)}(1-\lambda_1)^2 & \\ + \cdots + b_n(2n)(2n-1)\frac{\delta_{2n-2}(\lambda_n)}{\delta_{2n}(\lambda_n)}(1-\lambda_n)^2 &= (2n+1)^{(3)}, \\ \cdots & \\ b_1(2n)^{(2n-2)}\frac{\delta_2(\lambda_1)}{\delta_{2n}(\lambda_1)}(1-\lambda_1)^{2n-2} & \\ + \cdots + b_n(2n)^{(2n-2)}\frac{\delta_2(\lambda_n)}{\delta_{2n}(\lambda_n)}(1-\lambda_n)^{2n-2} &= (2n+1)^{(2n-1)}. \end{aligned} \tag{20}$$

Divide the k-th equation by $(2n)^{(2k-2)}$, subtract the k-th from the $(k+1)$st, and obtain

$$\sum_{j=1}^{n} b_j = 2n + 1,$$

$$\sum_{j=1}^{n} b_j \left[\frac{\delta_{2n-2}(\lambda_j)(1-\lambda_j)^2}{\delta_{2n}(\lambda_j)} - 1\right] = 0, \tag{21}$$

$$\cdots$$

$$\sum_{j=1}^{n} b_j \left[\frac{\delta_2(\lambda_j)(1-\lambda_j)^{2n-2} - \delta_4(\lambda_j)(1-\lambda_j)^{2n-4}}{\delta_{2n}(\lambda_j)}\right] = 0.$$

By (13) we have

$$\begin{aligned}\frac{\delta_{2k}(\lambda_j)(1-\lambda_j)^{2n-2k}}{\delta_{2n}(\lambda_j)} &= \frac{u_{k-1}(r_j)}{u_{n-1}(r_j)}(1/\lambda_j - 2 + \lambda_j)^{n-k} \\ &= (-2)^{n-k}(r_j+1)^{n-k}\frac{u_{k-1}(r_j)}{u_{n-1}(r_j)},\end{aligned}$$

so that

$$\frac{\delta_{2k}(\lambda_j)(1-\lambda_j)^{2n-2k}}{\delta_{2n}(\lambda_j)} - \frac{\delta_{2n+2}(\lambda_j)(1-\lambda_j)^{2n-2k-2}}{\delta_{2n}(\lambda_j)} = \frac{w_k(r_j)}{u_{n-1}(r_j)},$$

where $w_k(r)$ is a polynomial of degree $n-2$ in r. Set

$$\gamma_j = b_j/u_{n-1}(r_j).$$

Reverse the order of Eqs. (21). The first $n-1$ equations are now

$$\sum_{j=1}^{n} \gamma_j w_k(r_j) = 0 \qquad (k = 1, 2, \ldots, n-1).$$

Thus

$$\begin{aligned}\gamma_1 : \gamma_2 : \cdots : \gamma_n = &\begin{vmatrix} w_1(r_2) & \cdots & w_1(r_n) \\ \vdots & & \vdots \\ w_{n-1}(r_2) & \cdots & w_{n-1}(r_n)\end{vmatrix} \\ &: - \begin{vmatrix} w_1(r_1) & w_1(r_3) & \cdots & w_1(r_n) \\ \vdots & \vdots & & \vdots \\ w_{n-1}(r_1) & w_{n-1}(r_3) & \cdots & w_{n-1}(r_n)\end{vmatrix} \\ &: (-1)^{n-1}\begin{vmatrix} w_1(r_1) & w_1(r_2) & \cdots & w_1(r_{n-1}) \\ \vdots & \vdots & & \vdots \\ w_{n-1}(r_1) & w_{n-1}(r_2) & \cdots & w_{n-1}(r_{n-1})\end{vmatrix}.\end{aligned}$$

If we write

$$w_k(r) = w_{k,n-2}r^{n-2} + \cdots + w_{k,1}r + w_{k,0}\,,$$

then

$$\begin{vmatrix} w_1(r_2) & \cdots & w_1(r_n) \\ \vdots & & \vdots \\ w_{n-1}(r_2) & \cdots & w_{n-1}(r_n) \end{vmatrix} = \begin{vmatrix} w_{1,0} & \cdots & w_{1,n-2} \\ \vdots & & \vdots \\ w_{n-1,0} & \cdots & w_{n-1,n-2} \end{vmatrix} \cdot \begin{vmatrix} 1 & \cdots & 1 \\ r_2 & \cdots & r_n \\ \vdots & & \vdots \\ r_2^{n-2} & \cdots & r_n^{n-2} \end{vmatrix}$$

$$= W \cdot V(r_2, \ldots, r_n), \text{ etc.,} \tag{22}$$

where W is the first determinant on the right in (22) and $V(r_2, \ldots, r_n)$ is the Vandermonde determinant in the quantities $r_2, \ldots, r_n$, etc. Thus

$$(-1)^{j-1}\gamma_j = K \cdot V(r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_n), \ K \text{ a constant,}$$

and

$$b_j = K \cdot u_{n-1}(r_j)(-1)^{j-1}\, V(r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_n).$$

Substitution into the first of Eqs. (20) gives ($n \geqslant 1$):

$$K = (2n+1) \div \begin{vmatrix} u_{n-1}(r_1) & \cdots & u_{n-1}(r_n) \\ 1 & \cdots & 1 \\ r_1 & \cdots & r_n \\ \vdots & & \vdots \\ r_1^{n-2} & \cdots & r_n^{n-2} \end{vmatrix}$$

$$= (2n+1) \div [(-1)^{n-1}\, V(r_1, \ldots, r_n)],$$

since the polynomial $u_{n-1}(r)$ is monic. This gives

$$b_j = (2n+1)\frac{u_{n-1}(r_j)}{\prod_{k\neq j}(r_j - r_k)} = (2n+1)\frac{u_{n-1}(r_j)}{v_n'(r_j)}\,.$$

We now have

$$A(\theta) = (1-\theta)^{2n+1} + (2n+1)\sum_{j=1}^{n} \frac{u_{n-1}(r_j)}{\prod_{k\neq j}(r_j - r_k)}\, s_j(\theta), \qquad 0 \leqslant \theta \leqslant 1,$$

$$= (2n+1)\sum_{j=1}^{n} \lambda_j^{\,p} \frac{u_{n-1}(r_j)}{\prod_{k\neq j}(r_j - r_k)}\, s_j(\theta - p), \qquad 1 \leqslant p \leqslant \theta \leqslant p+1,$$

or, alternatively,

$$A(\theta) = (1-\theta)^{2n+1} + \sum_{j=1}^{n} b_j C_j(\theta), \qquad 0 \leqslant \theta \leqslant 1,$$

$$= \sum_{j=1}^{n} b_j C_j(\theta), \qquad 1 \leqslant \theta.$$

For the cubic, $b_1 = 3$ and

$$A(\theta) = (1-\theta)^3 + 3C_1(\theta).$$

The Cardinal Spline $A_N(\theta)$

The periodic cardinal spline $A_N(\theta)$ is of degree $2n+1$, has knots at the integer points, and satisfies

$$\begin{gathered} A_N(\theta) \in C^{2n}(-\infty, \infty), \qquad A_N(\theta + N) = A_N(\theta), \\ A_N(-\theta) = A_N(\theta), \quad A_N(0) = 1, \quad A_N(k) = 0 \qquad (k = 1, 2,..., N-1). \end{gathered} \tag{23}$$

On [0, 1], write

$$\begin{aligned} A_N(0) = (1-\theta)^{2n+1} &+ [a_1 s_1(\theta) + \cdots + a_n s_n(\theta)] \\ &+ a_{n+1}\lambda_n{}^N s_{n+1}(\theta) + \cdots + a_{2n}\lambda_1{}^N s_{2n}(\theta); \end{aligned}$$

for $1 \leqslant p \leqslant \theta \leqslant p+1 \leqslant N-1$,

$$\begin{aligned} A_N(\theta) = {} & [a_1\lambda_1{}^p s_1(\theta - p) + \cdots + a_n\lambda_n{}^p s_n(\theta - p)] \\ & + [a_{n+1}\lambda_n^{N-p} s_{n+1}(\theta - p) + \cdots + a_{2n}\lambda_1^{N-p} s_{2n}(\theta - p)]; \end{aligned} \tag{24'}$$

for $N-1 \leqslant \theta \leqslant N$,

$$\begin{aligned} A_N(\theta) = {} & (\theta - N + 1)^{2n+1} \\ & + [a_1\lambda_1^{N-1} s_1(\theta - N + 1) + \cdots + a_n\lambda_n^{N-1} s_n(\theta - N + 1)] \\ & + [a_{n+1}\lambda_n s_{n+1}(\theta - N + 1) + \cdots + a_{2n}\lambda_1 s_{2n}(\theta - N + 1)]. \end{aligned} \tag{24''}$$

The conditions (23a) require that

$$A_N^{(p)}(0) = A_N^{(p)}(N) \qquad (p = 1, 2,..., 2n). \tag{25}$$

These requirements are met for *even p*, in view of (15), if we set

$$b_j = b_{2n+1-j} .$$

For *p odd*, conditions (25) become, upon setting

$$\beta_j = (1 - \lambda_j^N)\, b_j ,$$

identical with Eqs. (20). Hence for $j = 1, 2,..., n$,

$$b_j = \frac{2n+1}{1-\lambda_j^N} \cdot \frac{u_{n-1}(r_j)}{\prod_{k\neq j}(r_j - r_k)} . \tag{26}$$

The representation (24), because of (15), now becomes

$$\begin{aligned} A_N(\theta) &= (1-\theta)^{2n+1} + \sum_{j=1}^{n} b_j[s_j(\theta) + \lambda_j^{N-1} s_j(1-\theta)], \qquad 0 \leqslant \theta \leqslant 1, \\ &= \sum_{j=1}^{n} b_j[\lambda_j^{p} s_j(\theta - p) + \lambda_j^{N-p-1} s_j(p+1-\theta)], \\ &\qquad\qquad 1 \leqslant p \leqslant \theta \leqslant p+1 \leqslant N-1, \quad (27) \\ &= (\theta - N + 1)^{2n+1} + \sum_{j=1}^{n} b_j[\lambda_j^{N-1} s_j(\theta - N + 1) + s_j(N-\theta)], \\ &\qquad\qquad N-1 \leqslant \theta \leqslant N. \end{aligned}$$

Splines of Interpolation—Periodic

Let us consider now the uniform mesh $\Delta : x_0 < x_j < \cdots < x_N$, $x_{k+1} = x_k + h(k = 0, 1,..., N-1)$. The periodic spline of interpolation, $S_\Delta(x)$, of degree $2n+1$ satisfying $S_\Delta(x_k) = f_k (k = 0, 1,..., N)$ is now simply

$$S_\Delta(x) = \sum_{k=1}^{N} f_k A_N\left(\frac{x - x_k}{h}\right).$$

From (27) it is seen that on $[x_p, x_{p+1}], 0 \leqslant p \leqslant N-1$, this becomes $(\xi = (x - x_p)/h)$

$$\begin{aligned}
&f_1 \sum_{j=1}^{n} b_j[\lambda_j^{p-1} s_j(\xi) + \lambda_j^{N-p} s_j(1-\xi)] \\
&\quad + \cdots + f_{p-1} \sum_{j=1}^{n} b_j[\lambda_j s_j(\xi) + \lambda_j^{N-2} s_j(1-\xi)] \\
&\quad + f_p \left\{\sum_{j=1}^{n} b_j[s_j(\xi) + \lambda_j^{N-1} s_j(1-\xi)] + (1-\xi)^{2n+1}\right\} \\
&\quad + f_{p+1} \left\{\xi^{2n+1} + \sum_{j=1}^{n} b_j[\lambda_j^{N-1} s_j(\xi) + s_j(1-\xi)]\right\} \\
&\quad + f_{p+2} \sum_{j=1}^{n} b_j[\lambda_j^{N-2} s_j(\xi) + \lambda_j s_j(1-\xi)] \\
&\quad + \cdots + f_N \sum_{j=1}^{n} b_j[\lambda_j{}^p s_j(\xi) + \lambda_j^{N-p} s_j(1-\xi)].
\end{aligned}$$

Set

$$\begin{aligned}
\sigma_{j,p} &= b_j f_0 \lambda_j{}^p + \cdots + f_{p-1}\lambda_j + f_p, \\
\tau_{j,p} &= b_j f_p + f_{p+1}\lambda_j + \cdots + f_N \lambda_j^{N-p}.
\end{aligned} \tag{28}$$

Thus

$$\begin{aligned}
\sigma_{j,0} &= b_j f_0, \qquad \sigma_{j,p} = \lambda_j \sigma_{j,p-1} + b_j f_p, \\
\tau_{j,N} &= b_j f_N, \qquad \tau_{j,p} = \lambda_j \sigma_{j,p+1} + b_j f_p,
\end{aligned} \tag{29}$$

and on $[x_p, x_{p+1}]$,

$$\begin{aligned}
S_\Delta(x) = \sum_{j=1}^{n} \{&s_j(\xi)[\sigma_{j,N-1}\lambda_j^{p+1} + \sigma_{j,p}(1 - \lambda_j{}^N)] \\
&+ s_j(1-\xi)[\tau_{j,1}\lambda_j^{N-p} + \tau_{j,p+1}(1 - \lambda_j{}^N)] \\
&+ (1-\xi)^{2n+1} f_p + \xi^{2n+1} f_{p+1}.
\end{aligned} \tag{30}$$

Once the quantities $\sigma_{j,p}$ and $\tau_{j,p}$ are constructed, $S_\Delta(x)$ is determined by the values of $s_j(\xi)$, $s_j(1-\xi)(j = 1,..., n)$, in the interval $0 \leqslant \xi \leqslant 1$. Reduction to the cubic case yields anew earlier results [1, Eq. (29)].

Splines of Interpolation—Non Periodic

Here, in addition to specifying $S_\Delta(x_p) = f_p$ ($p = 0, 1,..., N$), there are to be prescribed n end conditions at x_0 and at x_N. We restrict our attention to two types of such end conditions:

$$\text{Type I.} \quad S_\Delta^{(p)}(x_i) = f_i^{(p)} \qquad (i = 0, N;\ p = 1, 2,..., n).$$

$$\text{Type II.} \quad S_\Delta^{(p)}(x_i) = f_i^{(p)} \qquad (i = 0, N;\ p = n+1,..., 2n).$$

For our purpose it is required to construct bases of Type I and Type II for nonperiodic splines on the uniform mesh 0, 1,..., N. For Type I we require splines $B_{i,p}(\theta)(i = 0, N; p = 1, 2,..., n)$ satisfying

$$\begin{aligned} &B_{i,p}(k) = 0, \qquad k = 0, 1,..., N, \\ &B_{i,p}^{(q)}(0) = \delta_{i,0}\delta_{p,q}, \qquad B_{i,p}^{(q)}(N) = \delta_{i,N}\delta_{p,q} \qquad (q = 1, 2,..., n), \end{aligned} \tag{31}$$

for $i = 0, N, p = 1, 2,..., n$.

Consider the linear combination,

$$a_1C_1(\theta) + \cdots + a_nC_n(\theta) + a_{n+1}\lambda_1{}^NC_{2n}(\theta) + \cdots + a_{2n}\lambda_n{}^NC_{n+1}(\theta). \tag{32}$$

Set, for $j = 1,..., n$, $\alpha = 1,..., 2n$,

$$s_{\alpha j} = \frac{(-1)^{\alpha-1}}{(2n)^{(\alpha-1)}}C_j^{(\alpha)}(0) = \frac{(1-\lambda_j)^{\alpha-1}\,\delta_{2n+1-\alpha}(\lambda_j)}{\delta_{2n}(\lambda_j)}. \tag{33}$$

Then (32) becomes the $B_{i,p}(\theta)$ for Type I splines provided

$$\begin{aligned} &s_{11}a_1 + \cdots + s_{1n}a_n + \lambda_1{}^Ns_{11}a_{n+1} + \cdots + \lambda_n{}^Ns_{1n}a_{2n} = \delta_{i,0}\delta_{p,1}, \\ &s_{21}a_1 + \cdots + s_{2n}a_n - \lambda_1{}^Ns_{21}a_{n+1} - \cdots - \lambda_n{}^Ns_{2n}a_{2n} = -\delta_{2,0}\delta_{p,2}/(2n), \\ &\qquad\cdots \\ &s_{n1}a_1 + \cdots + s_{nn}a_n + (-1)^{n-1}\lambda_1{}^Ns_{n1}a_{n+1} + \cdots + (-1)^{n-1}\lambda_n{}^Ns_{nn}a_{2n} \\ &\qquad = (-1)^{n-1}\delta_{i,0}\delta_{p,n}/(2n)^{(n-1)}, \\ &\lambda_1{}^Ns_{11}a_1 + \cdots + \lambda_n{}^Ns_{1n}a_n + s_{11}a_{n+1} + \cdots + s_{1n}a_{2n} = \delta_{i,N}\delta_{p,1}, \\ &-\lambda_1{}^Ns_{21}a_1 - \cdots - \lambda_n{}^Ns_{2n}a_n + s_{21}a_{n+1} + \cdots + s_{2n}a_{2n} = \delta_{i,N}\delta_{p,2}/(2n), \\ &\qquad\cdots \\ &(-1)^{n-1}\lambda_1{}^Ns_{n1}a_1 + \cdots + (-1)^{n-1}\lambda_n{}^Ns_{nn}a_n + s_{n1}a_{n+1} + \cdots + s_{nn}a_n \\ &\qquad = \delta_{i,N}\delta_{p,n}/(2n)^{(n-1)}. \end{aligned} \tag{34}$$

The defining equations for Type II are

$$
\begin{aligned}
&s_{n+1,1}a_1 + \cdots + s_{n+1,n}a_n + (-1)^n \lambda_1^N s_{n+1,1}a_{n+1} + \cdots + (-1)^n \lambda_n^N s_{n+1,n}a_n \\
&\qquad = (-1)^n \delta_{i,0}\delta_{p,n+1}/(2n)^{(n)}, \\
&\qquad \cdots \\
&s_{2n,1}a_1 + \cdots + s_{2n,n}a_n - \lambda_1^N s_{2n,1}a_{n+1} - \cdots - \lambda_n^N s_{2n,n}a_{2n} \\
&\qquad = -\delta_{i,0}\delta_{p,2n}/(2n)^{(2n-1)}, \\
&(-1)^n \lambda_1^N s_{n+1,1}a_1 + \cdots + (-1)^n \lambda_n^N s_{n+1,n}a_n + s_{n+1,1}a_{n+1} + \cdots + s_{n+1,n}a_{2n} \\
&\qquad = \delta_{i,N}\delta_{p,n+1}/(2n)^{(n)}, \\
&\qquad \cdots \\
&-\lambda_1^N s_{2n,1}a_1 - \cdots - \lambda_n^N s_{2n,n}a_n + s_{2n,1}a_{n+1} + \cdots + s_{2n,n}a_{2n} \\
&\qquad = \delta_{i,N}\delta_{p,2n}/(2n)^{(2n-1)}.
\end{aligned} \tag{35}
$$

We rewrite these systems as

$$
\mathcal{A}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \mathcal{B}\begin{bmatrix} a_{n+1} \\ a_{n+2} \\ \vdots \\ a_{2n} \end{bmatrix} = \delta_{i,0}\mathcal{R}_1 , \tag{36}
$$

$$
\mathcal{B}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \mathcal{A}\begin{bmatrix} a_{n+1} \\ a_{n+2} \\ \vdots \\ a_{2n} \end{bmatrix} = \delta_{i,N}\mathcal{R}_2 , \tag{37}
$$

where, in the case of Type I splines,

$$
\mathcal{A} = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{bmatrix}, \qquad \mathcal{B} = \mathcal{W}\mathcal{A}\mathcal{L}, \tag{38}
$$

$$
\mathcal{R}_1 = \mathcal{W}\mathcal{R}_2 \equiv \mathcal{W}\begin{bmatrix} \delta_{p,1} \\ \delta_{p,2}/(2n) \\ \vdots \\ \delta_{p,n}/(2n)^{(n-1)} \end{bmatrix}, \tag{39}
$$

with

$$\mathscr{W} = \begin{bmatrix} +1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & (-1)^n \end{bmatrix}, \quad \mathscr{L} = \begin{bmatrix} \lambda_1{}^N & & & \\ & \lambda_2{}^N & & \\ & & \ddots & \\ & & & \lambda_n{}^N \end{bmatrix}. \tag{40}$$

In the case of Type II splines,

$$\mathscr{A} = \begin{bmatrix} s_{n+1,1} & \cdots & s_{n+1,n} \\ s_{n+2,1} & \cdots & s_{n+2,n} \\ \vdots & & \vdots \\ s_{2n,1} & \cdots & s_{2n,n} \end{bmatrix}, \qquad \mathscr{B} = (-1)^n \mathscr{W}\mathscr{A}\mathscr{L},$$

$$\mathscr{R}_1 = (-1)^n \mathscr{W}\mathscr{R}_2 = (-1)^n \mathscr{W} \begin{bmatrix} \delta_{p,n+1}/(2n)^{(n)} \\ \vdots \\ \delta_{p,2n}/(2n)^{(2n-1)} \end{bmatrix}.$$

The Type I spline is then

$$\sum_{k=0}^{N} f_k A(\theta - k) + \sum_{\alpha=1}^{n} \left(f_0^{(\alpha)} - \sum_{k=0}^{N} f_k A^{(\alpha)}(-k)\right) B_{0,\alpha}(\theta)$$
$$+ \sum_{\alpha=1}^{n} \left(f_N^{(\alpha)} - \sum_{k=0}^{N} f_k A^{(\alpha)}(N-k)\right) B_{N,\alpha}(\theta). \tag{41}$$

The Type II spline has similar form except that α ranges from $n + 1$ to $2n$.

We set forth the solutions of these systems for splines of degree 5 and 7 (splines of degree 3 are covered in Ref. [1]). For Type I splines,

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = (\mathscr{B}^{-1}\mathscr{A} - \mathscr{A}^{-1}\mathscr{B})^{-1} \{\delta_{i,0}\mathscr{B}^{-1}\mathscr{R}_1 - \delta_{i,N}\mathscr{A}^{-1}\mathscr{R}_2\}, \tag{42}$$

$$\begin{bmatrix} a_{n+1} \\ a_{n+2} \\ \vdots \\ a_{2n} \end{bmatrix} = (\mathscr{B}^{-1}\mathscr{A} - \mathscr{A}^{-1}\mathscr{B})^{-1} \{\delta_{i,N}\mathscr{B}^{-1}\mathscr{R}_2 - \delta_{i,0}\mathscr{A}^{-1}\mathscr{R}_1\}. \tag{43}$$

We have

$$(\mathscr{B}^{-1}\mathscr{A} - \mathscr{A}^{-1}\mathscr{B})^{-1} = (\mathscr{Q} - \mathscr{A}^{-1}\mathscr{B})(\mathscr{Q} + \mathscr{B}^{-1}\mathscr{A}),$$

where $\mathcal{Q}$ is the identity matrix. Inasmuch as $(\mathcal{Q} - \mathcal{A}^{-1}\mathcal{B})$ and $(\mathcal{Q} + \mathcal{B}^{-1}\mathcal{A})$ commute, we have

$$(\mathcal{B}^{-1}\mathcal{A} - \mathcal{A}^{-1}\mathcal{B})^{-1} = \tfrac{1}{2}\{(\mathcal{Q} + \mathcal{B}^{-1}\mathcal{A})^{-1} - (\mathcal{Q} - \mathcal{B}^{-1}\mathcal{A})^{-1}\}. \tag{44}$$

We set

$$\mathcal{A}^{-1} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}.$$

For convenience, we write t_j in place of λ_j^N so that, for the quintic spline,

$$\mathcal{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathcal{A} \begin{bmatrix} t_1 & \\ & t_2 \end{bmatrix}$$

and

$$\mathcal{B}^{-1}\mathcal{A} = \begin{bmatrix} t_1^{-1} & \\ & t_2^{-1} \end{bmatrix} \mathcal{A}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathcal{A}.$$

Now

$$\begin{aligned} \mathcal{A}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathcal{A} &= \begin{bmatrix} A_{11}s_{11} - A_{12}s_{21} & A_{11}s_{12} - A_{12}s_{22} \\ A_{21}s_{11} - A_{22}s_{21} & A_{21}s_{12} - A_{22}s_{22} \end{bmatrix} \\ &= \begin{bmatrix} 1 - 2A_{12}s_{21} & -2A_{12}s_{22} \\ -2A_{22}s_{21} & 1 - 2A_{22}s_{22} \end{bmatrix} \\ &= \mathcal{Q} - 2 \begin{bmatrix} A_{12}s_{21} & A_{12}s_{22} \\ A_{22}s_{21} & A_{22}s_{22} \end{bmatrix}, \end{aligned} \tag{45}$$

since

$$A_{i1}s_{1j} + A_{i2}s_{2j} = \delta_{ij}.$$

Further,

$$\begin{bmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{bmatrix}^2 = (a_1b_1 + a_2b_2) \begin{bmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{bmatrix}. \tag{46}$$

Thus,

$$\mathcal{B}^{-1}\mathcal{A} = \begin{bmatrix} t_1^{-1} & 0 \\ 0 & t_2^{-1} \end{bmatrix} \left\{ \mathcal{Q} - 2 \begin{bmatrix} A_{12}s_{21} & A_{12}s_{22} \\ A_{22}s_{21} & A_{22}s_{22} \end{bmatrix} \right\},$$

so that

$$\mathcal{Q} + \mathcal{B}^{-1}\mathcal{A} = \begin{bmatrix} \dfrac{1+t_1}{t_1} & 0 \\ 0 & \dfrac{1+t_2}{t_2} \end{bmatrix} \times \left\{ \mathcal{Q} - 2 \begin{bmatrix} \dfrac{1}{1+t_1} & 0 \\ 0 & \dfrac{1}{1+t_2} \end{bmatrix} \begin{bmatrix} A_{12}s_{21} & A_{12}s_{22} \\ A_{22}s_{21} & A_{22}s_{22} \end{bmatrix} \right\}$$

$$\equiv \begin{bmatrix} \dfrac{1+t_1}{t_1} & 0 \\ 0 & \dfrac{1+t_2}{t_2} \end{bmatrix} \{\mathcal{Q} - 2\mathcal{U}\}.$$

By (46),

$$\mathcal{U}^2 \equiv \begin{bmatrix} \dfrac{A_{12}}{1+t_1} s_{21} & \dfrac{A_{12}}{1+t_1} s_{22} \\ \dfrac{A_{22}}{1+t_2} s_{21} & \dfrac{A_{22}}{1+t_2} s_{22} \end{bmatrix}^2 = \alpha\mathcal{U}, \qquad \alpha = \frac{A_{12}s_{21}}{1+t_1} + \frac{A_{22}s_{22}}{1+t_2}, \tag{47}$$

so that

$$(\mathcal{Q} + \mathcal{B}^{-1}\mathcal{A})^{-1} = \{\mathcal{Q} + 2\mathcal{U} + 4\mathcal{U}^2 + \cdots\} \begin{bmatrix} \dfrac{t_1}{1+t_1} & 0 \\ 0 & \dfrac{t_2}{1+t_2} \end{bmatrix}$$

$$= \{\mathcal{Q} + \mathcal{U}(2 + 4\alpha + 8\alpha^2 + \cdots)\} \begin{bmatrix} \dfrac{t_1}{1+t_1} & 0 \\ 0 & \dfrac{t_2}{1+t_2} \end{bmatrix}$$

$$= \left\{\mathcal{Q} + \frac{2}{1-2\alpha}\mathcal{U}\right\} \begin{bmatrix} \dfrac{t_1}{1+t_1} & 0 \\ 0 & \dfrac{t_2}{1+t_2} \end{bmatrix}.$$

Similarly,

$$\mathcal{Q} - \mathcal{B}^{-1}\mathcal{A} = \begin{bmatrix} 1 - 1/t_1 & 0 \\ 0 & 1 - 1/t_2 \end{bmatrix} + 2 \begin{bmatrix} 1/t_1 & 0 \\ 0 & 1/t_2 \end{bmatrix} \begin{bmatrix} A_{12}s_{21} & A_{12}s_{22} \\ A_{22}s_{21} & A_{22}s_{22} \end{bmatrix}$$

$$= - \begin{bmatrix} \dfrac{1-t_1}{t_1} & 0 \\ 0 & \dfrac{1-t_2}{t_2} \end{bmatrix} \{\mathcal{Q} - 2\mathcal{V}\},$$

where

$$\mathscr{V} = \begin{bmatrix} \dfrac{1}{1-t_1} & 0 \\ 0 & \dfrac{1}{1-t_2} \end{bmatrix} \begin{bmatrix} A_{12}s_{21} & A_{12}s_{22} \\ A_{22}s_{21} & A_{22}s_{22} \end{bmatrix}.$$

Thus we obtain

$$(\mathscr{Q} - \mathscr{B}^{-1}\mathscr{A})^{-1} = -\left(\mathscr{Q} + \frac{2}{1-2\beta}\mathscr{V}\right) \begin{bmatrix} \dfrac{t_1}{1-t_1} & 0 \\ 0 & \dfrac{t_2}{1-t_2} \end{bmatrix},$$

using

$$\mathscr{V}^2 = \beta\mathscr{V}, \qquad \beta = \frac{A_{12}s_{21}}{1-t_1} + \frac{A_{22}s_{22}}{1-t_2}.$$

Hence

$$\begin{aligned}
&(\mathscr{B}^{-1}\mathscr{A} - \mathscr{A}^{-1}\mathscr{B})^{-1} \\
&= \frac{1}{2}\left\{\begin{bmatrix} \left(1 + \dfrac{2}{1-2\alpha}\dfrac{A_{12}s_{21}}{1+t_1}\right)\dfrac{t_1}{1+t_1} & \dfrac{2}{1-2\alpha}\dfrac{A_{12}s_{22}}{1+t_1}\dfrac{t_2}{1+t_2} \\ \dfrac{2}{1-2\alpha}\dfrac{A_{22}s_{21}}{1+t_2}\dfrac{t_1}{1+t_1} & \left(1 + \dfrac{2}{1-2\alpha}\dfrac{A_{22}s_{22}}{1+t_2}\right)\dfrac{t_2}{1+t_2} \end{bmatrix}\right. \\
&\quad \left. + \begin{bmatrix} \left(1 + \dfrac{2}{1-2\beta}\dfrac{A_{12}s_{21}}{1-t_1}\right)\dfrac{t_1}{1-t_1} & \dfrac{2}{1-2\beta}\dfrac{A_{12}s_{22}}{1-t_1}\dfrac{t_2}{1-t_2} \\ \dfrac{2}{1-2\beta}\dfrac{A_{22}s_{21}}{1-t_2}\dfrac{t_1}{1-t_1} & \left(1 + \dfrac{2}{1-2\beta}\dfrac{A_{22}s_{22}}{1-t_2}\right)\dfrac{t_2}{1-t_2} \end{bmatrix}\right\} \\
&= \frac{1}{(1-2\alpha)(1-2\beta)(1-t_1^2)(1-t_2^2)} \\
&\quad \times \begin{bmatrix} t_1(1-t_2^2)(1-2A_{12}s_{21}) & -2t_2A_{12}s_{22}(1-t_1t_2) \\ -2t_1A_{22}s_{21}(1-t_1t_2) & t_2(1-t_1^2)(1-2A_{22}s_{22}) \end{bmatrix}.
\end{aligned} \tag{48}$$

We may write

$$1 - 2\alpha = -\left(\frac{1-t_1}{1+t_1}A_{12}s_{21} + \frac{1-t_2}{1+t_2}A_{22}s_{22}\right),$$

$$1 - 2\beta = -\left(\frac{1+t_1}{1-t_1}A_{12}s_{21} + \frac{1+t_2}{1-t_2}A_{22}s_{22}\right),$$

$$(1-2\alpha)(1-2\beta) = 1 + \frac{4A_{12}s_{21}A_{22}s_{22}(t_1-t_2)^2}{(1-t_1^2)(1-t_2^2)}. \tag{49}$$

The quantities A_{ij}, s_{ij} are evaluated by means of the spline characteristic equation satisfied by $\lambda_j (j = 1, 2)$:

$$1 + 26\lambda + 66\lambda^2 + 26\lambda^3 + \lambda^4 = 0. \tag{50}$$

The roots are λ_1, λ_2, $1/\lambda_1$, $1/\lambda_2$, where

$$1/\lambda_1 < 1/\lambda_2 < -1 < \lambda_2 < \lambda_1 < 0.$$

For convenience, set $\mu_j = -1/\lambda_j$ so that $1 < \mu_2 < \mu_1$. By (50) written as

$$(\mu + 1/\mu)^2 - 26(\mu + 1/\mu) + 64 = 0,$$

we have

$$\mu_1 + 1/\mu_1 = 13 + \sqrt{105}, \qquad \mu_2 + 1/\mu_2 = 13 - \sqrt{105},$$

and hence

$$\sqrt{\mu_1} + 1/\sqrt{\mu_1} = \sqrt{15 + \sqrt{105}}, \qquad \sqrt{\mu_2} + 1/\sqrt{\mu_2} = \sqrt{15 - \sqrt{105}},$$

$$\sqrt{\mu_1} - 1/\sqrt{\mu_1} = \sqrt{11 + \sqrt{105}}, \qquad \sqrt{\mu_2} - 1/\sqrt{\mu_2} = \sqrt{11 - \sqrt{105}},$$

With

$$s_{11} = 1, \qquad s_{12} = 1,$$
$$s_{21} = (1 - \lambda_1)\, \delta_3(\lambda_1)/\delta_4(\lambda_1), \qquad s_{22} = (1 - \lambda_2)\, \delta_3(\lambda_2)/\delta_4(\lambda_2),$$

we have

$$A_{22} = 1/(s_{11}s_{22} - s_{12}s_{21}) = 1/(s_{22} - s_{21})$$

so that

$$\begin{aligned}
1/A_{22}s_{22} &= 1 - (s_{21}/s_{22}) \\
&= 1 - \frac{1 - \lambda_1}{1 + \lambda_1} \cdot \frac{1 + \lambda_2}{1 - \lambda_2} \cdot \frac{1 + 4\lambda_1 + \lambda_1^2}{1 + 4\lambda_2 + \lambda_2^2} \cdot \frac{1 + 10\lambda_2 + \lambda_2^2}{1 + 10\lambda_1 + \lambda_1^2} \\
&= 1 - \frac{\sqrt{\mu_1} + 1/\sqrt{\mu_1}}{\sqrt{\mu_1} - 1/\sqrt{\mu_1}} \cdot \frac{\sqrt{\mu_2} - 1/\sqrt{\mu_2}}{\sqrt{\mu_2} + 1/\sqrt{\mu_2}} \cdot \frac{\mu_1 + 1/\mu_1 - 4}{\mu_2 + 1/\mu_2 - 4} \\
&\quad \cdot \frac{\mu_2 + 1/\mu_2 - 10}{\mu_1 + 1/\mu_1 - 10} \\
&= 1 - \frac{23\sqrt{15} + 29\sqrt{7}}{32\sqrt{2}}.
\end{aligned}$$

Finally,

$$A_{12}s_{21} + A_{22}s_{22} = 1$$

and, since $A_{12} = -A_{22}$,

$$A_{12}s_{22} = -A_{22}s_{22}, \qquad A_{22}s_{21} = -A_{12}s_{21} = A_{22}s_{22} - 1.$$

For Type II splines,

$$\mathscr{A} = \begin{bmatrix} s_{31} & s_{32} \\ s_{41} & s_{42} \end{bmatrix} = \begin{bmatrix} \dfrac{(1-\lambda_1)^2\,\delta_2(\lambda_1)}{\delta_4(\lambda_1)} & \dfrac{(1-\lambda_2)^2\,\delta_2(\lambda_2)}{\delta_4(\lambda_2)} \\ \dfrac{(1-\lambda_1)^3}{\delta_4(\lambda_1)} & \dfrac{(1-\lambda_2)^3}{\delta_4(\lambda_2)} \end{bmatrix},$$

so that

$$\mathscr{A}^{-1} = \begin{bmatrix} A_{31} & A_{32} \\ A_{41} & A_{42} \end{bmatrix} = \frac{1}{2(\lambda_1 - \lambda_2)} \begin{bmatrix} \dfrac{(1-\lambda_2)\,\delta_4(\lambda_1)}{(1-\lambda_1)^2} & -\dfrac{(1+\lambda_2)\,\delta_4(\lambda_1)}{(1-\lambda_1)^2} \\ -\dfrac{(1-\lambda_1)\,\delta_4(\lambda_2)}{(1-\lambda_2)^2} & \dfrac{(1+\lambda_1)\,\delta_4(\lambda_2)}{(1-\lambda_2)^2} \end{bmatrix}.$$

Here we find

$$\begin{aligned}(\mathscr{B}^{-1}\mathscr{A} &- \mathscr{A}^{-1}\mathscr{B})^{-1} \\ &= \frac{1}{(1-2\alpha)(1-2\beta)(1-t_1^2)(1-t_2^2)} \\ &\quad \cdot \begin{bmatrix} t_1(1-t_2^2)(1-2A_{32}s_{41}) & -2t_2A_{32}s_{42}(1-t_1t_2) \\ -2t_1A_{42}s_{41}(1-t_1t_2) & t_2(1-t_1^2)(1-2A_{42}s_{42}) \end{bmatrix}, \end{aligned} \tag{51}$$

with

$$1 - 2\alpha = -\left(\frac{1-t_1}{1+t_1}A_{32}s_{41} + \frac{1-t_2}{1+t_2}A_{42}s_{42}\right),$$

$$1 - 2\beta = -\left(\frac{1+t_1}{1-t_1}A_{32}s_{41} + \frac{1+t_2}{1-t_2}A_{42}s_{42}\right).$$

Now

$$\frac{1}{A_{42}s_{42}} = \frac{(1+\lambda_1)(1-\lambda_2) - (1-\lambda_1)(1+\lambda_2)}{(1+\lambda_1)(1-\lambda_2)} = 1 - \frac{\sqrt{15}-\sqrt{7}}{2\sqrt{2}}.$$

Also

$$A_{32}s_{41} + A_{42}s_{42} = 1,$$

$$A_{32}/A_{42} = -(13 - \sqrt{105})/8, \qquad s_{41}/s_{42} = -(3\sqrt{15} + \sqrt{7})/8\sqrt{2}.$$

For the Type I spline of degree 7, a similar analysis gives

$$\mathscr{Q} + \mathscr{B}^{-1}\mathscr{A} = \begin{bmatrix} 1 + 1/t_1 & 0 & 0 \\ 0 & 1 + 1/t_2 & 0 \\ 0 & 0 & 1 + 1/t_3 \end{bmatrix} \{\mathscr{Q} - 2\mathscr{U}\},$$

where

$$\mathscr{U} = \begin{bmatrix} 1/(1 + t_1) & 0 & 0 \\ 0 & 1/(1 + t_2) & 0 \\ 0 & 0 & 1/(1 + t_3) \end{bmatrix} \cdot \begin{bmatrix} A_{12}s_{21} & A_{12}s_{22} & A_{12}s_{23} \\ A_{22}s_{21} & A_{22}s_{22} & A_{22}s_{23} \\ A_{32}s_{21} & A_{32}s_{22} & A_{32}s_{23} \end{bmatrix},$$

$$\mathscr{U}^2 = \alpha\mathscr{U},$$

$$\alpha = \frac{A_{12}s_{21}}{1 + t_1} + \frac{A_{22}s_{22}}{1 + t_2} + \frac{A_{32}s_{23}}{1 + t_3},$$

and

$$\mathscr{Q} - \mathscr{B}^{-1}\mathscr{A} = \begin{bmatrix} 1 - 1/t_1 & 0 & 0 \\ 0 & 1 - 1/t_2 & 0 \\ 0 & 0 & 1 - 1/t_3 \end{bmatrix} \{\mathscr{Q} - 2\mathscr{V}\},$$

where

$$\mathscr{V} = \begin{bmatrix} 1/(1 - t_1) & 0 & 0 \\ 0 & 1/(1 - t_2) & 0 \\ 0 & 0 & 1/(1 - t_3) \end{bmatrix} \cdot \begin{bmatrix} A_{12}s_{21} & A_{12}s_{22} & A_{12}s_{23} \\ A_{22}s_{21} & A_{22}s_{22} & A_{22}s_{23} \\ A_{32}s_{21} & A_{32}s_{22} & A_{32}s_{23} \end{bmatrix},$$

$$\mathscr{V}^2 = \beta\mathscr{V},$$

$$\beta = \frac{A_{12}s_{21}}{1 - t_1} + \frac{A_{22}s_{22}}{1 - t_2} + \frac{A_{32}s_{23}}{1 - t_3}.$$

Thus

$$(\mathscr{B}^{-1}\mathscr{A} - \mathscr{A}^{-1}\mathscr{B})^{-1}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} \left(1 + \frac{2}{1-2\alpha}\frac{A_{12}s_{21}}{1+t_1}\right)\frac{t_1}{1+t_1} & \frac{2}{1-2\alpha}\frac{A_{12}s_{22}}{1+t_1}\frac{t_2}{1+t_2} & \frac{2}{1-2\alpha}\frac{A_{12}s_{23}}{1+t_1}\frac{t_3}{1+t_3} \\ \frac{2}{1-2\alpha}\frac{A_{22}s_{21}}{1+t_2}\frac{t_1}{1+t_1} & \left(1 + \frac{2}{1-2\alpha}\frac{A_{22}s_{22}}{1+t_2}\right)\frac{t_2}{1+t_2} & \frac{2}{1-2\alpha}\frac{A_{22}s_{23}}{1+t_2}\frac{t_3}{1+t_3} \\ \frac{2}{1-2\alpha}\frac{A_{32}s_{21}}{1+t_3}\frac{t_1}{1+t_1} & \frac{2}{1-2\alpha}\frac{A_{32}s_{22}}{1+t_3}\frac{t_2}{1+t_2} & \left(1 + \frac{2}{1-2\alpha}\frac{A_{32}s_{23}}{1+t_3}\right)\frac{t_3}{1+t_3} \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} \left(1 + \frac{2}{1-2\beta}\frac{A_{12}s_{21}}{1-t_1}\right)\frac{t_1}{1-t_1} & \frac{2}{1-2\beta}\frac{A_{12}s_{22}}{1-t_1}\frac{t_2}{1-t_2} & \frac{2}{1-2\beta}\frac{A_{12}s_{23}}{1-t_1}\frac{t_3}{1-t_3} \\ \frac{2}{1-2\beta}\frac{A_{22}s_{21}}{1-t_2}\frac{t_1}{1-t_1} & \left(1 + \frac{2}{1-2\beta}\frac{A_{22}s_{22}}{1-t_2}\right)\frac{t_2}{1-t_2} & \frac{2}{1-2\beta}\frac{A_{22}s_{23}}{1-t_2}\frac{t_3}{1-t_3} \\ \frac{2}{1-2\beta}\frac{A_{32}s_{21}}{1-t_3}\frac{t_1}{1-t_1} & \frac{2}{1-2\beta}\frac{A_{32}s_{22}}{1-t_3}\frac{t_2}{1-t_2} & \left(1 + \frac{2}{1-2\beta}\frac{A_{32}s_{23}}{1-t_3}\right)\frac{t_3}{1-t_3} \end{bmatrix}\right\}$$

$$= \frac{1}{(1-2\alpha)(1-2\beta)}$$

$$\times \begin{bmatrix} \frac{t_1}{1-t_1^2}\left[1 - 2A_{12}s_{21} + 4\frac{A_{22}s_{22}A_{32}s_{23}(t_3-t_2)^2}{(1-t_2^2)(1-t_3^2)}\right], \\ \frac{-2t_1A_{22}s_{21}}{(1-t_1^2)(1-t_2^2)}\left[1 - t_1t_2 + 2\frac{(t_3-t_1)(t_3-t_2)}{1-t_3^2}A_{32}s_{23}\right], \\ \frac{-2t_1A_{32}s_{21}}{(1-t_1^2)(1-t_3^2)}\left[1 - t_1t_3 + 2\frac{(t_2-t_1)(t_2-t_3)}{1-t_2^2}A_{22}s_{22}\right], \end{bmatrix}$$

$$\frac{-2t_2A_{12}s_{22}}{(1-t_1^2)(1-t_2^2)}\left[1-t_1t_2+2\frac{(t_3-t_1)(t_3-t_2)}{1-t_3^2}A_{32}s_{23}\right],$$

$$\frac{t_2}{1-t_2^2}\left[1-2A_{22}s_{22}+4\frac{A_{12}s_{21}A_{32}s_{23}(t_3-t_1)^2}{(1-t_3^2)(1-t_1^2)}\right],$$

$$\frac{-2t_2A_{32}s_{22}}{(1-t_2^2)(1-t_3^2)}\left[1-t_1t_2+2\frac{(t_1-t_2)(t_1-t_3)}{1-t_1^2}A_{12}s_{21}\right],$$

$$\left.\begin{matrix}\frac{-2t_3A_{12}s_{23}}{(1-t_1^2)(1-t_3^2)}\left[1-t_1t_2+2\frac{(t_2-t_3)(t_2-t_1)}{1-t_2^2}A_{22}s_{22}\right]\\ \frac{-2t_3A_{22}s_{23}}{(1-t_2^2)(1-t_3^2)}\left[1-t_1t_2+2\frac{(t_1-t_2)(t_1-t_3)}{1-t_1^2}A_{12}s_{21}\right]\\ \frac{t_3}{1-t_3^2}\left[1-2A_{32}s_{23}+4\frac{A_{12}s_{21}A_{22}s_{22}(t_2-t_1)^2}{(1-t_1^2)(1-t_2^2)}\right]\end{matrix}\right]. \quad (52)$$

Here

$$1-2\alpha=-\left[\frac{1-t_1}{1+t_1}A_{12}s_{21}+\frac{1-t_2}{1+t_2}A_{22}s_{22}+\frac{1-t_3}{1+t_3}A_{32}s_{23}\right],$$

$$1-2\beta=-\left[\frac{1+t_1}{1-t_1}A_{12}s_{21}+\frac{1+t_2}{1-t_2}A_{22}s_{22}+\frac{1+t_3}{1-t_3}A_{32}s_{23}\right],$$

and

$$(1-2\alpha)(1-2\beta)=1+4\left[\frac{A_{12}s_{21}A_{22}s_{22}(t_1-t_2)^2}{(1-t_1^2)(1-t_2^2)}+\frac{A_{12}s_{21}A_{32}s_{23}(t_1-t_3)^2}{(1-t_1^2)(1-t_3^2)}\right.$$
$$\left.+\frac{A_{22}s_{22}A_{32}s_{23}(t_2-t_3)^2}{(1-t_2^2)(1-t_3^2)}\right].$$

The matrix $\mathcal{A}$ is

$$\begin{bmatrix}1 & 1 & 1\\ \frac{(1-\lambda_1)\,\delta_5(\lambda_1)}{\delta_6(\lambda_1)} & \frac{(1-\lambda_2)\,\delta_5(\lambda_2)}{\delta_6(\lambda_2)} & \frac{(1-\lambda_3)\,\delta_5(\lambda_3)}{\delta_6(\lambda_3)}\\ \frac{(1-\lambda_1)^2\,\delta_4(\lambda_1)}{\delta_6(\lambda_1)} & \frac{(1-\lambda_2)^2\,\delta_4(\lambda_2)}{\delta_6(\lambda_2)} & \frac{(1-\lambda_3)^2\,\delta_4(\lambda_3)}{\delta_6(\lambda_3)}\end{bmatrix}.$$

Set

$$R_j=\frac{(1-\lambda_j)\,\delta_5(\lambda_j)}{\delta_6(\lambda_j)}\begin{vmatrix}1 & 1\\ \frac{(1-\lambda_{j-1})^2\,\delta_4(\lambda_{j-1})}{\delta_6(\lambda_{j-1})} & \frac{(1-\lambda_{j+1})^2\,\delta_4(\lambda_{j+1})}{\delta_6(\lambda_{j+1})}\end{vmatrix} \quad (j=1,2,3) \quad (53)$$

where λ_0 is λ_3 and λ_4 is λ_1.
Then the determinant of $\mathcal{A}$ is

$$|\mathcal{A}|=R_1+R_2+R_3,$$

and

$$\begin{aligned} 1/A_{12}s_{21} &= 1 + (R_2/R_1) + (R_3/R_1), \\ 1/A_{22}s_{22} &= 1 + (R_1/R_2) + (R_3/R_2), \\ 1/A_{32}s_{23} &= 1 + (R_1/R_3) + (R_2/R_3). \end{aligned}$$

We find that, with $2\theta_j = (\mu_j + 1/\mu_j) = -(\lambda_j + 1/\lambda_j)$,

$$\frac{R_2}{R_1} = -\sqrt{\frac{(\theta_2+1)(\theta_1-1)}{(\theta_1+1)(\theta_2-1)}}\,\frac{\theta_2^2 - 13\theta_2 + 16}{\theta_1^2 - 13\theta_1 + 16}\,\frac{\theta_3-\theta_1}{\theta_3-\theta_2} \times \frac{4\theta_1\theta_3 - 11(\theta_1+\theta_3) + 64}{4\theta_2\theta_3 - 11(\theta_2+\theta_3) + 64},$$

and similarly for R_1/R_3 and R_2/R_3. Further,

$$\frac{A_{12}}{A_{22}} = -\frac{\theta_1^2 - 28\theta_1 + 61}{\theta_2^2 - 28\theta_2 + 61}\cdot\frac{\theta_3-\theta_2}{\theta_3-\theta_1}\cdot\frac{4\theta_2\theta_3 - 11(\theta_2+\theta_3)+64}{4\theta_1\theta_3 - 11(\theta_1+\theta_3)+64},$$

with a similar expression for A_{32}/A_{22}. Finally,

$$\frac{s_{21}}{s_{22}} = \sqrt{\frac{(\theta_1+1)(\theta_2-1)}{(\theta_1-1)(\theta_2+1)}}\,\frac{(\theta_1^2 - 13\theta_1+16)(\theta_2^2-28\theta_2+61)}{(\theta_2^2-13\theta_2+16)(\theta_1^2-28\theta_1+61)},$$

with a similar expression for s_{23}/s_{22}. Thus the terms $A_{i2}s_{2j}$ $(i, j = 1, 2, 3)$ may be determined.

For Type II splines of degree 7, the inverse matrix $(\mathscr{B}^{-1}\mathfrak{A} - \mathfrak{A}^{-1}\mathscr{B})^{-1}$ is obtained by replacing $A_{i2}s_{2j}$ by $A_{i+3,2}s_{5,j}$ in (52). The matrix $\mathfrak{A}$ is here

$$\mathfrak{A} = \begin{bmatrix} \frac{(1-\lambda_1)^3\,\delta_3(\lambda_1)}{\delta_6(\lambda_1)} & \frac{(1-\lambda_2)^3\,\delta_3(\lambda_2)}{\delta_6(\lambda_2)} & \frac{(1-\lambda_3)^3\,\delta_3(\lambda_3)}{\delta_6(\lambda_3)} \\ \frac{(1-\lambda_1)^4\,\delta_2(\lambda_1)}{\delta_6(\lambda_1)} & \frac{(1-\lambda_2)^4\,\delta_2(\lambda_2)}{\delta_6(\lambda_2)} & \frac{(1-\lambda_3)^4\,\delta_3(\lambda_3)}{\delta_6(\lambda_3)} \\ \frac{(1-\lambda_1)^5}{\delta_6(\lambda_1)} & \frac{(1-\lambda_2)^5}{\delta_6(\lambda_2)} & \frac{(1-\lambda_3)^5}{\delta_6(\lambda_3)} \end{bmatrix}.$$

The quantity R_j in (53) becomes

$$R_j = \frac{(1-\lambda_j)^4\,\delta_2(\lambda_j)}{\delta_6(\lambda_j)} \begin{vmatrix} \frac{(1-\lambda_{j-1})^3\,\delta_3(\lambda_{j-1})}{\delta_6(\lambda_{j-1})} & \frac{(1-\lambda_{j+1})^3\,\delta_3(\lambda_{j+1})}{\delta_6(\lambda_{j+1})} \\ \frac{(1-\lambda_{j-1})^5}{\delta_6(\lambda_{j-1})} & \frac{(1-\lambda_{j+1})^5}{\delta_6(\lambda_{j+1})} \end{vmatrix}.$$

Here

$$R_2/R_1 = -[(\theta_2^2 - 1)/(\theta_1^2 - 1)]^{1/2} [(\theta_3 - \theta_1)/(\theta_3 - \theta_2)],$$

$$\frac{A_{42}}{A_{52}} = -\sqrt{\frac{(\theta_1 - 1)(\theta_2 + 1)}{(\theta_1 + 1)(\theta_2 - 1)}} \cdot \frac{\theta_2 + 1}{\theta_1 + 1} \cdot \frac{\theta_3 - \theta_2}{\theta_3 - \theta_1} \frac{\theta_1^2 - 28\theta_1 + 61}{\theta_2^2 - 28\theta_2 + 61},$$

$$s_{51}/s_{52} = (\theta_2^2 - 28\theta_2 + 61)(\theta_1 + 1)^2/(\theta_1^2 - 28\theta_1 + 61)(\theta_2 + 1)^2.$$

The following table 1 is included to facilitate the computation of the polynomial splines on uniform meshes.

TABLE I

Degree $(2n + 1)$	Roots $\lambda_j (j = 1,..., n)$	$-\theta_j = (\lambda_j + \lambda_j^{-1})/2$
3	−0.267949	2.0
5	−0.043096	11.623475
	−0.430575	1.376525
7	−0.009148	54.657179
	−0.122555	4.141091
	−0.535280	1.201730
9	−0.002121	235.704814
	−0.043223	11.589632
	−0.201751	2.579184
	−0.607997	1.126371
11	−0.000511	979.321811
	−0.016670	30.003007
	−0.089760	5.615315
	−0.272180	1.973107
	−0.661266	1.086758
13	−0.000125	3996.799003
	−0.006738	74.209020
	−0.043214	11.591967
	−0.138901	3.669134
	−0.333108	1.667572
	−0.701894	1.063305
15	−0.000031	16162.566757
	−0.002802	178.499424
	−0.021752	22.997199
	−0.075905	6.624909
	−0.186520	2.773936
	−0.385586	1.489522
	−0.733873	1.048254

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